# ARITHMETIC AND GEOMETRY OVER LOCAL FIELDS

BRUNO ANGLÈS AND TUAN NGO DAC

Arithmetic geometry is a very active area of mathematics, with important and deep connections to various areas such as algebraic geometry, number theory and Lie theory.

The goal of this volume aims to introduce graduate students and young researchers to some of recent research topics in arithmetic geometry over local fields. The lectures are centered around two common themes: the study of Drinfeld modules and non-Archimedean analytic geometry.

The notes of this volume grow out from the lectures given during the research program "Arithmetic and geometry of local and global fields" which took place at the Vietnam Institute of Advanced Study in Mathematics (VIASM) from June to August 2018. Two of them were given in the occasion of VIASM School on Number Theory (see https://hanoi-nt18.sciencesconf.org/). The others were presented as advanced courses during the research seminar.

The authors, all leading experts in the subject, have made a great effort to make the notes as self-contained as possible. In addition to introducing the basic tools, the lectures aim to present an overview of recent developments in the arithmetic and geometry of local fields and related topics. The included examples and suggested concrete research problems will enable young researchers to quickly reach the frontiers of this fascinating branch of mathematics.

## Contents of this volume

The volume consists of seven lectures:

(1) *Some Elements on Berthelot's Arithmetic* $\mathfrak{D}$*-Modules* by **Daniel Caro** (University of Caen Normandy, France),
(2) *Difference Galois Theory for the "Applied" Mathematician* by **Lucia Di Vizio** (CNRS and University of Versailles Saint Quentin, France),
(3) *Igusa's Conjecture on Exponential Sums Modulo p m and Local-Global Principle* by **Nguyen Huu Kien** (KU Leuven, Belgium),
(4) *From the Carlitz Exponential to Drinfeld Modular Forms* by **Federico Pellarin** (Institute Camille Jordan and University of Saint-Etienne, France),
(5) *Berkovich Curves and Schottky Uniformization I: The Berkovich Affine Line* by **Jérôme Poineau** (University of Caen Normandy, France) and **Daniele Turchetti** (Dalhousie University, Canada),
(6) *Berkovich Curves and Schottky Uniformization II: Analytic Uniformization of Mumford Curves* by **Jérôme Poineau** (University of Caen Normandy, France) and **Daniele Turchetti** (Dalhousie University, Canada),

*Date*: October 2020.

(7) *On the Stark Units of Drinfeld Modules* by **Floric Tavares Ribeiro** (University of Caen Normandy, France).

The first lecture by D. Caro offers an introduction to $p$-differential methods in arithmetic geometry. First, he reviews Berthelot's ring of $p$-adic differential operators, which plays an important role in the theory of arithmetic $\mathfrak{D}$-modules. Next, he extends it to some finite level on $p$-adic formal affine smooth schemes. Finally, concrete examples and a guide to further reading are also provided. The material assumes a basic knowledge of ring theory and algebraic geometry.

The second lecture by L. Di Vizio gives an overview of the Galois theory of difference equations. The first part presents a guide to the key definitions and results of difference Galois theory. In the second part, interesting applications to transcendence and differential transcendence are treated in detail. Note that the framework is the same as that of Papanikolas' theory in the setting of Drinfeld modules. The curious reader may wish to refer to the lectures of F. Pellarin and F. Tavares Ribeiro for more details.

The third lecture by K. Nguyen is a survey on Igusa's conjecture around exponential sums motivated by the study of local-global principles for forms of higher degree. After introducing the notion of exponential sums and those modulo $p^m$ with some examples, he formulates a coarse form of Igusa's conjecture on a uniform bound of those exponential sums and explains its relations with Igusa's local zeta functions, the monodromy conjecture, and fiber integrals. He then states Igusa's conjecture on exponential sums and gives an overview of recent progress on this conjecture, in particular the most recent breakthrough of Cluckers, Mustaţă, and the author. The lecture ends with a general picture of the local-global principle for forms and the contribution of the aforementioned conjecture in this direction.

The fourth lecture by F. Pellarin presents a friendly introduction to the theory of Drinfeld modular forms attached to the affine line over a finite field. Drinfeld modular forms in positive characteristic are defined as analogues of classical modular forms by the pioneering works of Goss and Gekeler. The notes gradually introduce the very first basic elements of the arithmetic theory of Drinfeld modules, then the Drinfeld upper-half plane and its topology, and end with Drinfeld modular forms. The key notions are illustrated with many examples. The lecture also contains several advanced parts such as Drinfeld modular forms with values in BanachPreface vii algebras. These course notes will enable the reader to gently enter into this rich and still developing theory.

The self-contained survey by J. Poineau and D. Turchetti consists of two lectures on non-Archimedean curves and Schottky uniformization from the point of view of Berkovich geometry. The first part, the fifth lecture, could be read as an elementary course on the theory of Berkovich spaces with emphasis on the affine line. The authors introduce basic definitions and properties with full details and proofs. The second part, the sixth lecture, is more advanced and deals with the theory of uniformization of curves under Berkovich's theory. After introducing the notion of Mumford curves and Schottky groups, the authors present an analytic proof of Schottky uniformization. Many examples, explicit research problems, and a guide for further reading are also provided. The reader who is interested in Schottky groups in the language of rigid analytic spaces is invited to read the relevant parts of the lecture of F. Pellarin.

The last lecture by F. Tavares Ribeiro presents some recent developments in the arithmetic theory of the special values of Goss zeta functions. This lecture is an exposition on Stark units of Drinfeld modules over the ring of polynomials over a finite field. The notes are also the occasion to introduce basic definitions of Drinfeld modules and more recent fundamental objects linked to an analytic class number formula obtained by L. Taelman: $L$-values, unit modules, and class modules attached to a Drinfeld module. The author then presents the notion of Stark units and gives their basic properties. Finally, he gives several applications of Stark units, in particular, to the study of congruence properties of Bernoulli–Carlitz numbers. He also gives hints for a general base ring.

**Acknowledgments**

We would like to acknowledge the many people whose help was essential for this volume.

We would like to thank the speakers and the authors, Daniel Caro, Lucia Di Vizio, Nguyen Huu Kien, Federico Pellarin, Jérôme Poineau, Floric Tavares Ribeiro, Lenny Taelman and Daniele Turchetti, for their enthusiastic investment to this semester and this volume. We also thank all the referees for many valuable comments and suggestions that help improving the exposition of each lecture as well as the whole volume.

We would like to express our gratitude to the directors, the researchers and administrative staff of the VIASM for the hospitality and excellent working conditions during the special semester. Special thanks are due to Lê Minh Ha, Ngô Bao-Châu, Phung Hô Hai, Lê Thi Lan Anh and Nguyên Hoang Anh.

Our semester was partially supported by the VIASM, the Institute of Mathematics in Hanoi (IMH), the European Research Council (ERC Starting Grant TOSSIBERG), the ANR Grant PerCoLaTor (ANR-14-CE25-0002-01), the Institut Henri Poincaré (IHP), the Foundation Compositio Mathematica and the GDR Structuration de la théorie des nombres (CNRS).

Université de Caen Normandie, Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139, Campus II, Boulevard Maréchal Juin, B.P. 5186, 14032 Caen Cedex, France.

*Email address*: `bruno.angles@unicaen.fr`

CNRS - Université Claude Bernard Lyon 1, Institut Camille Jordan, UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

*Email address*: `ngodac@math.univ-lyon1.fr`